# A note of deriving sea surface height equations

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Start with

$$\int_{z}^{0} \frac{\partial \epsilon(z')}{\partial x} dz' \tag{1}$$

where $\epsilon$ is density anomaly. When doing vertically integration over the whole water column $H$, it becomes,

$$\int_{-H}^{0} \int_{z}^{0} \frac{\partial \epsilon(z')}{\partial x} dz' dz \tag{2}$$

where equation (1) can be regraded as a function of $z$, say $F(z)$, so equation (2) becomes

$$\int_{-H}^{0} F(z) dz \tag{3}$$

Paper says this can be done by integration by parts, but be careful. Here is how to evaluated, instructed by Dima. By integration by parts, we will have

$$\int u dv = uv - \int v du \tag{4}$$

Therefore, equation (3) can be seen as

$$u = F(z) \tag{5}$$

$$dv = dz \tag{6}$$

$$v = z \tag{7}$$

So equation (3) can be done as

$$
\begin{aligned}
zF(z)|_{-H}^{0} - \int_{-H}^{0} z\frac{\partial F(z)}{\partial z}dz \\
&= 0F(0) - (-H)F(-H) - \int_{-H}^{0} z\frac{\partial F(z)}{\partial z}dz \\
&= H\int_{-H}^{0} \frac{\partial \epsilon(z')}{\partial x}dz' - \int_{-H}^{0} z\frac{\partial F(z)}{\partial z}dz \qquad (8)
\end{aligned}
$$

The second term of equation (8) needs to deal with the derivative of $F(z)$. Recall that integral with variable limits

$$
\frac{\partial}{\partial x}\int_{a}^{b} f(x,z)dz = f(b)\frac{\partial b}{\partial x} - f(a)\frac{\partial a}{\partial x} + \int_{a}^{b} \frac{\partial f(x,z)}{\partial x}dz \qquad (9)
$$

We can proceed,

$$
\begin{aligned}
\frac{\partial F(z)}{\partial z} &= \frac{\partial}{\partial z}\left(\int_{z}^{0} \frac{\partial \epsilon(z')}{\partial x}dz'\right) \\
&= \frac{\partial \epsilon(0)}{\partial x}\frac{\partial 0}{\partial z} - \frac{\partial \epsilon(z)}{\partial x}\frac{\partial z}{\partial z} + \int_{z}^{0} \frac{\partial}{\partial z}\left(\frac{\partial \epsilon(z')}{\partial x}\right)dz' \qquad (10)
\end{aligned}
$$

Note that there is no dependent variables for differentiation with respect to $z$ for the third term of equation (10), so only the second term will be come out. Finally, equation (8) becomes

$$
\begin{aligned}
H\int_{-H}^{0} \frac{\partial \epsilon(z')}{\partial x}dz' - \int_{-H}^{0} z\frac{\partial F(z)}{\partial z}dz &= H\int_{-H}^{0} \frac{\partial \epsilon(z')}{\partial x}dz' + \int_{-H}^{0} z\frac{\partial \epsilon(z)}{\partial x}dz \\
&= \int_{-H}^{0} (H+z)\frac{\partial \epsilon(z)}{\partial x}dz \qquad (11)
\end{aligned}
$$

This is the equation found in paper Liu and Wesiberg (2007) and Csanady (1979).