A Review of Leif Thomas's *Formation of intrathermocline eddies at ocean fronts by wind-driven destruction of potential vorticity*

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Contents

1 Prerequisite knowledge

1.1 Ertel potential vorticity

Before we start to understand what destruction of potential vorticity (PV) is, we need to formulate the vorticity equation and know how it can be related to the Ertel PV. First, the vector form of the equation of motions is

$$\frac{\partial \mathbf{u}}{\partial t} + (2\boldsymbol{\Omega} + \boldsymbol{\omega}) \times \mathbf{u} = -\frac{1}{\rho} \bigtriangledown p + \bigtriangledown \left\{ \Phi - \frac{|\mathbf{u}|^2}{2} \right\} + \nu \bigtriangledown^2 \mathbf{u} \tag{1}$$

where $\boldsymbol{\omega} = \bigtriangledown \times \mathbf{u}$ and both second terms on LHS and RHS come from $(\mathbf{u} \cdot \bigtriangledown)\mathbf{u} = \boldsymbol{\omega} \times \mathbf{u} + \bigtriangledown^2 \mathbf{u}$. This originates from the vector idensity that $\bigtriangledown \times \mathbf{u} \times \mathbf{u} = \boldsymbol{\omega} \times \mathbf{u} = (\mathbf{u} \cdot \bigtriangledown)\mathbf{u} - \bigtriangledown \left(\frac{|\mathbf{u}|^2}{2} \right)$. Here the bold symbol represents vector so that $\mathbf{u}(u, v, w)$ is vecolity vector; $\boldsymbol{\Omega}$ is the earth rotation and $\boldsymbol{\omega}$ defines as the vorticity. The remnant parts in (1) are p for pressure, ρ for density, Φ for geopotential and ν for viscocity coefficient.

Taking curl of (1) we immediately form the equation for vortivity:

$$\frac{\partial}{\partial t} \bigtriangledown \times \mathbf{u} + \bigtriangledown \times \{(2\boldsymbol{\Omega} + \boldsymbol{\omega}) \times \mathbf{u}\} = -\bigtriangledown \times \left(\frac{1}{\rho} \bigtriangledown p \right) + \nu \bigtriangledown \times (\bigtriangledown^2 \mathbf{u}) + \bigtriangledown \times \bigtriangledown \left\{ \Phi - \frac{|\mathbf{u}|^2}{2} \right\} \tag{2}$$

If we define the absolute vorticity,

$$\boldsymbol{\omega_a} = 2\boldsymbol{\Omega} + \boldsymbol{\omega} \tag{3}$$

and use the following vector identities,

$$\bigtriangledown \times \bigtriangledown(\phi) = 0 \text{ if } \phi \text{ is a scalar} \tag{4}$$

$$\begin{aligned} \bigtriangledown^2 (\bigtriangledown \times \mathbf{u}) &= \bigtriangledown \times (\bigtriangledown^2 \mathbf{u}) \\ &= \bigtriangledown^2 \boldsymbol{\omega} \end{aligned} \tag{5}$$

$$\begin{aligned} -\bigtriangledown \times \left(\frac{1}{\rho} \bigtriangledown p \right) &= -\left[\bigtriangledown \left(\frac{1}{\rho} \right) \times \bigtriangledown p + \frac{1}{\rho} \bigtriangledown \times \bigtriangledown p \right] \\ &= -\left[-\frac{\bigtriangledown \rho}{\rho^2} \times \bigtriangledown p \right] \\ &= \frac{\bigtriangledown \rho \times \bigtriangledown p}{\rho^2} \end{aligned} \tag{6}$$

then (2) becomes

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \bigtriangledown \times (\boldsymbol{\omega_a} \times \mathbf{u}) = \frac{\bigtriangledown \rho \times \bigtriangledown p}{\rho^2} + \nu \bigtriangledown^2 \omega \tag{7}$$

Note that the vector identity, $\bigtriangledown \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\bigtriangledown \cdot \mathbf{B}) - \mathbf{B}(\bigtriangledown \cdot \mathbf{A}) + (\mathbf{B} \cdot \bigtriangledown)\mathbf{A} - (\mathbf{A} \cdot \bigtriangledown)\mathbf{B}$, so that the second term on LHS of (7) can be shown as

$$\begin{aligned}\bigtriangledown \times (\boldsymbol{\omega_a} \times \mathbf{u}) &= \boldsymbol{\omega_a}(\bigtriangledown \cdot \mathbf{u}) - \mathbf{u}(\bigtriangledown \cdot \boldsymbol{\omega_a}) + (\mathbf{u} \cdot \bigtriangledown)\boldsymbol{\omega_a} - (\boldsymbol{\omega_a} \cdot \bigtriangledown)\mathbf{u} \\ &= \boldsymbol{\omega}_i^{\boldsymbol{a}} \frac{\partial \mathbf{u}_j}{\partial x_j} + \mathbf{u}_j \frac{\partial \boldsymbol{\omega}_i^{\boldsymbol{a}}}{\partial x_j} - \mathbf{u}_i \frac{\partial \boldsymbol{\omega}_j^{\boldsymbol{a}}}{\partial x_j} - \boldsymbol{\omega}_j^{\boldsymbol{a}} \frac{\partial \mathbf{u}_i}{\partial x_j} \end{aligned} \tag{8}$$

The second term on LHS of (8) is zero due to the divergence of a curl field is 0. Then (7) can be written as

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u} \cdot \bigtriangledown)\boldsymbol{\omega_a} - (\boldsymbol{\omega_a} \cdot \bigtriangledown)\mathbf{u} + \boldsymbol{\omega_a}(\bigtriangledown \cdot \mathbf{u}) = \frac{\bigtriangledown \rho \times \bigtriangledown p}{\rho^2} + \nu \bigtriangledown^2 \boldsymbol{\omega} \tag{9}$$

If we treat the earth rotation is time invariant or its temporal evolution can be neglected in our formulation, we can come up with

$$\frac{d\boldsymbol{\omega_a}}{dt} = (\boldsymbol{\omega_a} \cdot \bigtriangledown)\mathbf{u} - \boldsymbol{\omega_a}(\bigtriangledown \cdot \mathbf{u}) + \frac{\bigtriangledown \rho \times \bigtriangledown p}{\rho^2} + \nu \bigtriangledown^2 \boldsymbol{\omega} \tag{10}$$

This is the so-called vorticity equation under the presence of baroclinicity and friction. Before we proceed, let's take a look on the first two terms on RHS of (10),

$$\begin{aligned}(\boldsymbol{\omega_a} \cdot \bigtriangledown)\mathbf{u} - \boldsymbol{\omega_a}(\bigtriangledown \cdot \mathbf{u}) &= \boldsymbol{\omega_a} \frac{\partial}{\partial z} \left[u\hat{i} + v\hat{j} + w\hat{k}\right] - \boldsymbol{\omega_a}\hat{k} \left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right] \\ &= \boldsymbol{\omega_a} \frac{\partial u}{\partial z}\hat{i} + \boldsymbol{\omega_a} \frac{\partial v}{\partial z}\hat{j} - \boldsymbol{\omega_a} \left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right] \hat{k} \end{aligned} \tag{11}$$

Here if we treat the rate of change of absolute vorticity (see (10)) is purely attributed to $-\boldsymbol{\omega_a} \left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right] \hat{k}$, the physical meaning of this term can be illustrated by the following figure (**Fig. 1**).

Imaging that we have two areas $S1$ and $S2$, in which the boundaries of $S2$ is formed by a velocity field extends toward positive x and y directions with respect to $S1$ during a time interval $\triangle t$. We can write down,

$$\begin{aligned} S1 &= \triangle x \triangle y \\ S2 &= \triangle x \triangle y \left(1 + \frac{\partial u}{\partial x}\triangle t\right)\left(1 + \frac{\partial v}{\partial y}\triangle t\right) \\ &\approx \triangle x \triangle y \left(1 + \frac{\partial u}{\partial x}\triangle t + \frac{\partial v}{\partial y}\triangle t\right) \end{aligned} \tag{12}$$

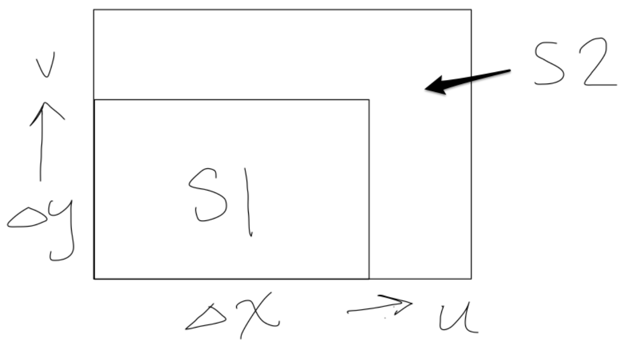

Figure 1: A schematic of two areas $S1$ and $S2$. The area of $S2$ is formed by a velocity field with respect to $S1$.

Therefore the rate of change of unit volume

$$\lim_{t \to 0} \frac{S2 - S1}{\triangle t} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \tag{13}$$

From (13), we can say that

$$\begin{aligned} \frac{dA}{dt} &= \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) A \\ \Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= \frac{1}{A} \frac{dA}{dt} \end{aligned} \tag{14}$$

where A is the cross-section area of an imaginary tube. Therefore, we can express $-\boldsymbol{\omega_a} \left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right]$ as $-\frac{\boldsymbol{\omega_a}}{A} \frac{dA}{dt}$, then (10) becomes

$$\begin{aligned} \frac{d\boldsymbol{\omega_a}}{dt} &= -\frac{\boldsymbol{\omega_a}}{A} \frac{dA}{dt} \\ \frac{d\boldsymbol{\omega_a}}{dt} + \frac{\boldsymbol{\omega_a}}{A} \frac{dA}{dt} &= 0 \\ \Rightarrow \frac{d}{dt} \left(\boldsymbol{\omega_a} A \right) &= 0 \end{aligned} \tag{15}$$

This result tells us that the physical meaning of $-\boldsymbol{\omega_a} \left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right] \hat{k}$ is for vortex stretching, in which the larger the cross-section area in the vortex tube the smaller the resulting absolute vorticity and vice versa.

Now if the rate of change of absolute vorticity is induced by $\boldsymbol{\omega_a}\frac{\partial u}{\partial z}\hat{\imath}$, we can interpret this term via the following diagram (**Fig. 2**).

$$\begin{aligned}
&\frac{d\boldsymbol{\omega_a}}{dt} = \boldsymbol{\omega_a}\frac{\partial u}{\partial z} \\
\Rightarrow &\frac{d\boldsymbol{\omega_a}}{\boldsymbol{\omega_a}} = \frac{\partial u}{\partial z}dt \\
\Rightarrow &\frac{\triangle\boldsymbol{\omega_a}}{\boldsymbol{\omega_a}} = \frac{\triangle u}{\triangle z}\triangle t = \frac{\triangle x}{\triangle z} = tan(\gamma)
\end{aligned} \tag{16}$$

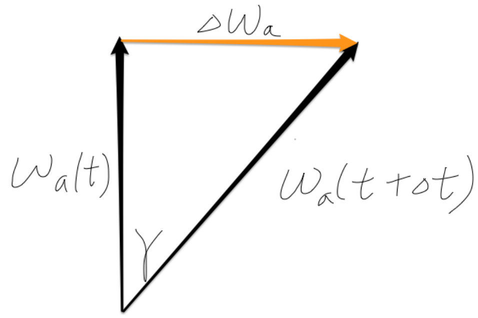

Figure 2: A schematic of tilting of vorticity axis under the presence of current shear.

Therefore, under the presence of current shear, the axis of absolute vorticity will be tilted by an angle γ. We conclude the vorticity equation in (10) that the rate of change of absolute vorticity is from 1) tilting of the vortex tube, 2) stretching of the vortex tube, 3) baroclinic production of vorticity and 4) viscous diffusion of vorticity.

If the thermodynamic change can be neglected, then $\frac{d\rho}{dt} \approx 0$, the divergence term can be ignored in (10). We can write (10) as

$$\frac{d}{dt}\left(\frac{\boldsymbol{\omega_j^a}}{\rho}\right) = \frac{\boldsymbol{\omega_j^a}}{\rho}\frac{\partial u_i}{\partial x_j} + \frac{1}{\rho^3}\frac{\partial \rho}{\partial x_j}\frac{\partial p}{\partial x_k} + \frac{\nu}{\rho}\bigtriangledown^2\boldsymbol{\omega}_i \tag{17}$$

If we define

$$\frac{d\lambda}{dt} = S \tag{18}$$

where λ is an unknown property and S is its source(sink) term. Now we proceed

$$
\begin{aligned}
\boldsymbol{\omega_a} \cdot \frac{d}{dt} \bigtriangledown \lambda &= \boldsymbol{\omega}_i^{\boldsymbol{a}} \frac{d}{dt}\frac{\partial \lambda}{\partial x_i} \\
&= \boldsymbol{\omega}_i^{\boldsymbol{a}} \left[\frac{\partial}{\partial t} + u_j \frac{\partial}{\partial j}\right] \frac{\partial \lambda}{\partial x_i} \\
&= \boldsymbol{\omega}_i^{\boldsymbol{a}} \frac{\partial}{\partial x_i}\frac{\partial \lambda}{\partial t} + \boldsymbol{\omega}_i^{\boldsymbol{a}} \frac{\partial}{\partial x_i}\left[u_j \frac{\partial \lambda}{\partial x_j}\right] - \boldsymbol{\omega}_i^{\boldsymbol{a}} \frac{\partial u_j}{\partial x_i}\frac{\partial \lambda}{\partial x_j} \\
&= \boldsymbol{\omega}_i^{\boldsymbol{a}} \frac{\partial}{\partial x_i}\frac{d\lambda}{dt} - \boldsymbol{\omega}_i^{\boldsymbol{a}} \frac{\partial u_j}{\partial x_i}\frac{\partial \lambda}{\partial x_j} \\
&= \boldsymbol{\omega}_i^{\boldsymbol{a}} \frac{\partial S}{\partial x_i} - \boldsymbol{\omega}_i^{\boldsymbol{a}} \frac{\partial u_j}{\partial x_i}\frac{\partial \lambda}{\partial x_j} \qquad (19)
\end{aligned}
$$

We find that

$$
\frac{\boldsymbol{\omega_a}}{\rho} \cdot \frac{d}{dt} \bigtriangledown \lambda = \frac{\boldsymbol{\omega_a}}{\rho} \cdot \bigtriangledown S - \left[\left(\frac{\boldsymbol{\omega_a}}{\rho} \cdot \bigtriangledown\right) \mathbf{u}\right] \cdot \bigtriangledown \lambda \qquad (20)
$$

Then if we do $\bigtriangledown\lambda \cdot \frac{d}{dt}\left(\frac{\boldsymbol{\omega_a}}{\rho}\right)$, it is

$$
\bigtriangledown\lambda \cdot \frac{d}{dt}\left(\frac{\boldsymbol{\omega_a}}{\rho}\right) = \left(\frac{\boldsymbol{\omega_a}}{\rho} \cdot \bigtriangledown\right) \mathbf{u} \cdot \bigtriangledown\lambda + \frac{1}{\rho^3} \bigtriangledown \lambda \cdot [\bigtriangledown\rho \times \bigtriangledown p] + \frac{\nu}{\rho} \bigtriangledown \lambda \cdot \bigtriangledown^2 \boldsymbol{\omega} \quad (21)
$$

If we put (20) and (21) together,

$$
\begin{aligned}
\frac{\boldsymbol{\omega_a}}{\rho} \cdot \frac{d}{dt} \bigtriangledown \lambda + \bigtriangledown\lambda \cdot \frac{d}{dt}\left(\frac{\boldsymbol{\omega_a}}{\rho}\right) &= \frac{d}{dt}\left(\frac{\boldsymbol{\omega_a}}{\rho} \cdot \bigtriangledown\lambda\right) \\
&= \frac{1}{\rho^3} \bigtriangledown \lambda \cdot [\bigtriangledown\rho \times \bigtriangledown p] + \frac{\boldsymbol{\omega_a}}{\rho} \cdot \bigtriangledown S + \frac{\nu}{\rho} \bigtriangledown \lambda \cdot \bigtriangledown^2 \boldsymbol{\omega} \qquad (22)
\end{aligned}
$$

If λ is a conservative term, saying potential density, then $\frac{d\lambda}{dt} = 0 = S$; and inviscid ($\nu = 0$); and barotropic ($\bigtriangledown\rho \times \bigtriangledown p = 0$) or $\lambda = \lambda(\rho, p)$, we will have

$$
\frac{d}{dt}\left(\frac{\boldsymbol{\omega_a}}{\rho} \cdot \bigtriangledown\lambda\right) = 0 \qquad (23)
$$

showing that $q = \frac{\boldsymbol{\omega_a}}{\rho} \cdot \bigtriangledown\lambda$ is conserved. This quantity is defined as the Ertel potential vorticity (PV). When we review Leif Thomas's paper (T08, [6]) later, q is first quantify we will see and that is why we need to know this term first. Next we are going to derive the flux form of Ertel PV equation and review the impermeability theorem, which will be used throughout T08.

1.2 The impermeability theorem

1.2.1 The flux form of Ertel PV equation

The notations in this section will follow Marshall et al's 2001 JPO paper (M01, [2]). First we introduce the flux form of Ertel PV equation.

$$\frac{\partial}{\partial t}(\rho Q) + \bigtriangledown \cdot \mathbf{J} = 0 \tag{24}$$

$$\text{where } Q = -\frac{1}{\rho}\boldsymbol{\omega_a} \cdot \bigtriangledown\sigma \tag{25}$$

where σ is the potential density. Note that (25) is the Ertel PV we have just showed in the last section. The term $\mathbf{J}$ is the flux of Q and is called the $\mathbf{J}$ vector. There are three important properties in (24): 1) *No matter what thermodynamic variable is chosen, there is always a flux form of PV equation*, 2) *it must be 0 on RHS of* (24) and 3) ***J** vectors can not pass through σ surfaces. σ surfaces are impermeable to Ertel PV.*

If we proceed

$$\begin{aligned}\frac{\partial}{\partial t}(\rho Q) &= \frac{\partial}{\partial t}(-\boldsymbol{\omega_a} \cdot \bigtriangledown\sigma) \\ &= -\frac{\partial}{\partial t} \bigtriangledown \cdot(\boldsymbol{\omega_a}\sigma) \\ &= - \bigtriangledown \cdot\mathbf{j}\end{aligned} \tag{26}$$

where $\mathbf{j}$ is defined as $\frac{\partial}{\partial t}(\boldsymbol{\omega_a}\sigma)$ and this is true that the conservation of ρQ can be expressed by the divergence of vectors. The next question is what the physical meaning of $\mathbf{j}$ vectors is.

Expanding $\mathbf{j}$,

$$\begin{aligned}\mathbf{j} &= \frac{\partial}{\partial t}(\boldsymbol{\omega_a}\sigma) \\ &= \boldsymbol{\omega_a}\frac{\partial\sigma}{\partial t} + \sigma\frac{\partial\boldsymbol{\omega_a}}{\partial t} \\ &= \boldsymbol{\omega_a}\frac{\partial\sigma}{\partial t} + \sigma\frac{\partial}{\partial t}(2\boldsymbol{\Omega} + \bigtriangledown \times \mathbf{u}) \\ &= \boldsymbol{\omega_a}\frac{\partial\sigma}{\partial t} + [\bigtriangledown \times (\sigma\mathbf{u})] - \frac{\partial}{\partial t}(\bigtriangledown\sigma \times \mathbf{u}) - \bigtriangledown \times \mathbf{u}\frac{\partial\sigma}{\partial t} \\ &= \boldsymbol{\omega_a}\frac{\partial\sigma}{\partial t} + \frac{\partial\mathbf{u}}{\partial t} \times \bigtriangledown\sigma + \bigtriangledown \times \left(\sigma\frac{\partial\mathbf{u}}{\partial t}\right)\end{aligned} \tag{27}$$

It is found that the third term on RHS of (27) does not provide contribution for PV flux, this term is non-divergent. The problem is to determine a non-divergent gauge $\mathbf{X}$ such that the flux $\mathbf{J}$,

Now the equation of motions (where it is very similar to (1) but here we use M01's notations)

$$\frac{\partial \mathbf{u}}{\partial t} = -\boldsymbol{\omega_a} \times \mathbf{u} - \bigtriangledown \left(\frac{|\mathbf{u}|^2}{2} + \frac{p}{\rho_0} \right) - \frac{\rho^{'}}{\rho} \bigtriangledown \Phi + \mathbf{F} \tag{28}$$

where $\mathbf{F}$ is the frictional force. Then we use

$$\frac{\rho^{'}}{\rho} \bigtriangledown \Phi = \bigtriangledown \left(\frac{\rho^{'}}{\rho} \Phi \right) - \frac{\Phi}{\rho_0} \bigtriangledown \rho^{'} \tag{29}$$

it shows that (28) becomes

$$\frac{\partial \mathbf{u}}{\partial t} = -\boldsymbol{\omega_a} \times \mathbf{u} - \bigtriangledown \left(\frac{|\mathbf{u}|^2}{2} + \frac{p}{\rho_0} + \frac{\rho^{'}}{\rho} \Phi \right) + \frac{\rho^{'}}{\rho} \bigtriangledown \rho^{'} + \mathbf{F} \tag{30}$$

Here we define $M = \frac{p}{\rho_0} + \frac{\rho^{'}}{\rho} \Phi$ as Montgomery potential and $\pi = M + \frac{|\mathbf{u}|^2}{2}$ as Bernoulli function, then take a cross product of (30) with $\bigtriangledown \sigma$ we have

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} \times \bigtriangledown \sigma &= -\left(\boldsymbol{\omega_a} \times \mathbf{u} \right) \times \bigtriangledown \sigma - \bigtriangledown \pi \times \bigtriangledown \sigma + \frac{\Phi}{\rho_0} \bigtriangledown \rho^{'} \times \bigtriangledown \sigma + \mathbf{F} \times \bigtriangledown \sigma \\ \Rightarrow \left(\frac{\partial \mathbf{u}}{\partial t} + \bigtriangledown \pi \right) \times \bigtriangledown \sigma &= \boldsymbol{\omega_a} \left(\bigtriangledown \sigma \cdot \mathbf{u} \right) - \mathbf{u} \left(\bigtriangledown \sigma \cdot \boldsymbol{\omega_a} \right) + \frac{\Phi}{\rho_0} \bigtriangledown \rho^{'} \times \bigtriangledown \sigma + \mathbf{F} \times \bigtriangledown \sigma \end{aligned} \tag{31}$$

where we apply the identity that $-\left(\boldsymbol{\omega_a} \times \mathbf{u} \right) \times \bigtriangledown \sigma = \bigtriangledown \sigma \times \left(\boldsymbol{\omega_a} \times \mathbf{u} \right) = \boldsymbol{\omega_a} \left(\bigtriangledown \sigma \cdot \mathbf{u} \right) - \mathbf{u} \left(\bigtriangledown \sigma \cdot \boldsymbol{\omega_a} \right)$.

Notice that $-\rho \mathbf{u} \left(\frac{1}{\rho} \bigtriangledown \sigma \cdot \boldsymbol{\omega_a} \right) = \rho \left(-\frac{1}{\rho} \boldsymbol{\omega_a} \cdot \bigtriangledown \sigma \right) \mathbf{u} = \rho Q \mathbf{u}$. If adding $\boldsymbol{\omega_a} \frac{\partial \sigma}{\partial t}$ on both sides of (31) and compare with (27) we will have

$$\begin{aligned} \boldsymbol{\omega_a} \frac{\partial \sigma}{\partial t} + \left(\frac{\partial \mathbf{u}}{\partial t} + \bigtriangledown \pi \right) \times \bigtriangledown \sigma &= \rho Q \mathbf{u} + \boldsymbol{\omega_a} \frac{d \sigma}{dt} + \mathbf{F} \times \bigtriangledown \sigma + \frac{\Phi}{\rho_0} \bigtriangledown \rho^{'} \times \bigtriangledown \sigma \\ &= \mathbf{J} \end{aligned} \tag{32}$$

Here the non-divergent gauge $\mathbf{X} = \bigtriangledown \pi \times \bigtriangledown \sigma$ and the $\mathbf{J}$ vector are determined. For terms on LHS in (32) are further emphasized in M01. For terms on RHS in (32) are used in Marshall and Nurser's 1992 JPO paper (M92, [3]). We are going to use the terms on RHS in (32) to reveal the impermeability theorem.

1.2.2 The impermeability theorem

Following M92's notations, and define $B = -g\frac{d\sigma}{dt}$ and $\rho^{'} = \rho^{'}(\rho^{'}, \sigma)$ then (24) becomes

$$\frac{\partial}{\partial t}(\rho Q) + \bigtriangledown \cdot \left(\rho Q\mathbf{u} - \frac{1}{g}B\boldsymbol{\omega_a} + \mathbf{F} \times \bigtriangledown\sigma\right) = 0 \tag{33}$$

Manipulating a bit we can get

$$\begin{aligned}
\frac{\partial}{\partial t}(\rho Q) + \rho Q \bigtriangledown \cdot\mathbf{u} + \mathbf{u} \cdot \bigtriangledown\rho Q - \frac{1}{g}B \bigtriangledown \cdot\boldsymbol{\omega_a} - \frac{1}{g}\boldsymbol{\omega_a} \bigtriangledown \cdot B \\
+ \bigtriangledown \rho \cdot (\bigtriangledown \times \mathbf{F}) - \mathbf{F} \cdot \bigtriangledown \times \bigtriangledown\sigma = 0 \\
\Rightarrow \frac{\partial}{\partial t}(\rho Q) + \mathbf{u} \cdot \bigtriangledown\rho Q = \frac{1}{g}\boldsymbol{\omega_a} \cdot \bigtriangledown B - \bigtriangledown \times \mathbf{F} \cdot \bigtriangledown\rho \\
\Rightarrow \frac{dQ}{dt} = -\frac{1}{\rho} \bigtriangledown \cdot N_Q \\
\text{where } N_Q = -\frac{1}{g}B\boldsymbol{\omega_a} + \mathbf{F} \times \bigtriangledown\sigma
\end{aligned} \tag{34}$$

This is obvious that why N_Q is called the non-advective term and we will see it again in T08. The important aspect of (34) is that the rate of change of PV can be expressed by the divergence of non-advective terms. We can conclude that the change of PV comes from 1) gradients of buoyancy forcing in the direction of $\boldsymbol{\omega_a}$ and 2) curl of frictional forces that lie on surfaces of constant σ (isopycnals).

Under a background current $\mathbf{U}$, if there is a controlling volume V which contains two isopycnal surfaces (θ_1 and θ_2) and a bounding surface ∂V_s lies between them (**Fig. 3**), and we define that the rate of change of θ within this domain is 0. Then we have

$$\begin{aligned}
\frac{\partial\theta}{\partial t} + \mathbf{U} \cdot \bigtriangledown\theta = \frac{\partial\theta}{\partial t} + \mathbf{u} \cdot \bigtriangledown\theta + \mathbf{U} \cdot \bigtriangledown\theta - \mathbf{u} \cdot \bigtriangledown\theta = 0 \\
\Rightarrow \frac{d\theta}{dt} + (\mathbf{U} - \mathbf{u}) \cdot \bigtriangledown\theta = 0
\end{aligned} \tag{35}$$

Integrating (33) for all V,

$$\begin{aligned}
\frac{d}{dt}\iiint_V \rho Q dV + \iiint_V \bigtriangledown \cdot \left((\mathbf{u} - \mathbf{U})\,\rho Q - -\frac{1}{g}B\boldsymbol{\omega_a} + \mathbf{F} \times \bigtriangledown\theta\right) dV = 0 \\
\Rightarrow \frac{d}{dt}\iiint_V \rho Q dV + \iint_S \left((\mathbf{u} - \mathbf{U})\,\rho Q - \frac{1}{g}B\boldsymbol{\omega_a} + \mathbf{F} \times \bigtriangledown\theta\right) \cdot \mathbf{n}dS = 0
\end{aligned} \tag{36}$$

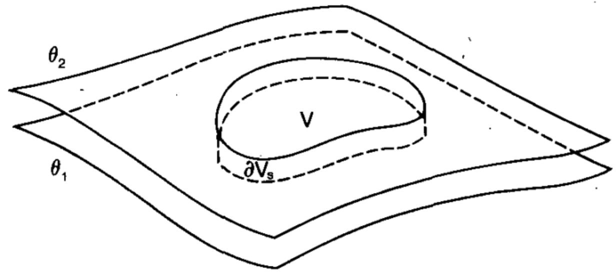

Figure 3: A schematic of controlling volume bound by two isopycnal surfaces θ_1 and θ_2 (Adapted from Haynes and McIntyre, 1987, [1]).

where the divergence theorem is used. Geometrically, $\mathbf{n}$ and $\bigtriangledown\theta$ are on the same plane, so that $\mathbf{F}\times\bigtriangledown\theta\cdot\mathbf{n}=0$. And from (35) we can get

$$\begin{aligned}\left((\mathbf{u}-\mathbf{U})\,\rho Q-\frac{1}{g}B\boldsymbol{\omega_a}\right)\cdot\mathbf{n}&=\left((\mathbf{u}-\mathbf{U})\,\boldsymbol{\omega_a}\cdot\bigtriangledown\theta-\frac{1}{g}B\boldsymbol{\omega_a}\right)\cdot\mathbf{n}\\&=-\boldsymbol{\omega_a}\left((\mathbf{U}-\mathbf{u})\cdot\bigtriangledown\theta+\frac{1}{g}B\right)\cdot\mathbf{n}\\&=0\end{aligned}\tag{37}$$

This indicates that PV can propagate in a velocity normal to the isopycnal surfaces but makes no contribution to the flux of PV. Summarizing, the impermeability theorem states that there is no flux of PV *across* isopycnal surfaces.

2 Review of the paper

2.1 General concept

In the beginning of the content, the author introduced the phenomenon called intrathermocline eddies (ITE) or submesoscale coherent vortices with weak stratification in the central core and a bulge-shaped isopycnal surfaces (**Fig. 4**). This configuration makes it contain anticyclonic vorticity. Their length scale is small compared with the first baroclinic Rossby radius of deformation. Due to the nature of anticyclonic vorticity, its PV will be low in the core region. Therefore, the existence of low PV water will be related to the generation of ITE. The main idea of this paper is the formation of low PV water via upward flux of PV through the interaction between wind stress and oceanic frontal structure.

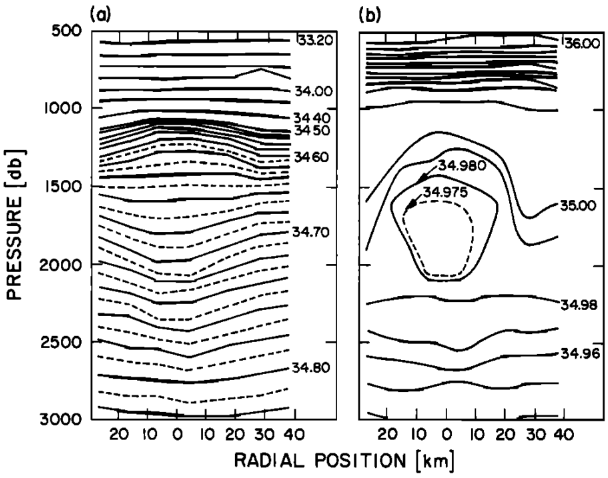

Fig. 4. Contour plots of (*a*) potential density (σ_{1500}) and (*b*) salinity (per mil) across the LDE subthermocline SCV [*Elliot and Sanford*, 1985*a*, Figures 7 and 8].

Figure 4: An example of intrathermocline eddy (Adapted from McWilliams, 1985, [4]).

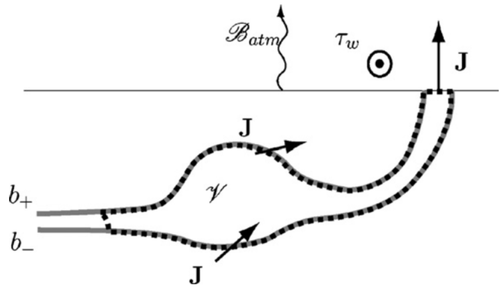

Figure 5: A schematic of outcropping frontal structure with a blowing down-front wind (into the page) (Adapted from Thomas, 2008).

When we look at the PV equation expressed as flux form shown in the paper, they are very similar to what we have derived in (33). Following this idea and using the controlling volume used in the paper (**Fig. 5**), the impermeability theorem indicates that the only way to lose PV is through the outcropping area of the front. One source of this upward PV flux can be seen from the term $\frac{1}{g}B\boldsymbol{\omega_a}$ in (33). Since it is upward, the vertical component of $\boldsymbol{\omega_a}$ is projected to the vertical buoyancy shear. That is why in the paper the upward PV flux is defined as

$$J_z^D = -\zeta_{abs} D \tag{38}$$

where $\zeta_{abs} = 2\mathbf{\Omega} + \hat{k} \cdot \bigtriangledown \times \mathbf{u}$ and $D = \frac{\partial \beta}{\partial z}$ (β is the air-sea flux)

The other source of the upward PV flux is the term $\mathbf{F} \times \bigtriangledown\sigma$ in (33), which can be seen that if the surface wind stress has a curl product associated with horizontal density gradient. Neglect the sign and this is what we see in the paper the upward PV flux associated with frictional forces,

$$J_z^F = \bigtriangledown_h b \times \mathbf{F} \tag{39}$$

where $b = -g\rho/\rho_0$. Applying the thermal wind relation,

$$\bigtriangledown_h b = f \frac{\partial \mathbf{u}_g}{\partial z} \times \hat{k} \tag{40}$$

where $f = 2\Omega$ and $\mathbf{u}_g$ is the geostrophic current then (39) becomes

$$J_z^F = \bigtriangledown_h b \times \mathbf{F} = \left(f \frac{\partial \mathbf{u}_g}{\partial z} \cdot \mathbf{F} \right) \hat{k} \tag{41}$$

Again it shows that this flux is upward if the frictional force is in the direction of the geostrophic shear. Summarizing, there are two components associated with the upward PV flux out of the fluid. One is associated with the buoyancy loss to the atmospheric forcing (eg., reducing stratification and enhancing mixing, see (38)), and the other is related to the wind stress and the current shear (41). The focus here is the down-front wind stress. These induce PV destruction and therefore provide a basis for generation of ITE.

2.2 The role of winds

The other concept is that the wind-driven flow redistributes PV via wind-driven buoyancy flux,

$$\beta_{wind} = \mathbf{M}_E \cdot \bigtriangledown_h b \tag{42}$$

where $\mathbf{M}_E$ is the wind-driven Ekman transport. In his another paper ([5]), he indicated that secondary ageostrophic circulations (ASCs) will also occur. The mechanisms mentioned previously indicate that the down-front wind stress translates heavier water out of the front into the ambient relatively fresh water. Due to the upward PV flux through the outcropping area, these translated water is inherently PV-low and are then dragged down from the surface by ASCs. ASCs also upwell PV-high water from below to the surface and these water are subsequently lose PV via the upward PV flux (38, 41). This process, while PV is extracted from the front continuously and stratification still exists, forms a layer of low PV and becomes a signature of ITE.

The goal of this paper is to use numerical experiments to delineate this formation process of ITE. Since PV is defined as

$$q = \boldsymbol{\omega_a} \cdot \bigtriangledown b \tag{43}$$

Note that this definition is still the same as what we have shown in (25). Then we decompose q into two components, one is the vertical component q_{vert} and the other is for horizontal one q_{bc}.

$$\begin{aligned} q_{vert} &= \boldsymbol{\omega_a}\hat{k} \cdot \bigtriangledown b\hat{k} \\ &= \zeta_{abs} N^2 \end{aligned} \tag{44}$$

$$\begin{aligned} q_{bc} &= \boldsymbol{\omega_a}(\hat{\imath}, \hat{\jmath}) \cdot \bigtriangledown_h b(\hat{\imath}, \hat{\jmath}) \\ &= \frac{\partial u}{\partial z}\frac{\partial b}{\partial y} - \frac{\partial v}{\partial z}\frac{\partial b}{\partial x} \end{aligned} \tag{45}$$

where $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ are neglected. Using (40), we can find

$$q_{bc} = -f \left| \frac{\partial \mathbf{u}_g}{\partial z} \right|^2 \leq 0 \tag{46}$$

where it means that we are dealing with a geostrophic flow and the presence of horizontal buoyancy gradient (baroclinicity) always leads to *negative* contribution on PV.

If $q \leq 0$, let $\zeta = \hat{k} \cdot \bigtriangledown \times \mathbf{u}$, then

$$\begin{aligned} q_{vert} + q_{bc} &\leq 0 \\ \Rightarrow \zeta_{abs} N^2 + f \left| \frac{\partial \mathbf{u}_g}{\partial z} \right|^2 &\leq 0 \\ \Rightarrow (f + \zeta) \frac{N^2}{\left| \frac{\partial \mathbf{u}_g}{\partial z} \right|^2} - f &\leq 0 \\ \Rightarrow \left(1 + \frac{\zeta}{f}\right) \frac{N^2}{\left| \frac{\partial \mathbf{u}_g}{\partial z} \right|^2} &\leq 1 \end{aligned} \tag{47}$$

This is related to Richardson number that baroclinic flow with strong current shear results in low PV and is referred to as baroclinically low PV. The following numerical experiment highlights this process in a three-dimensional aspect.

2.3 Results of numerical experiments

The simulation shows that the down-front wind stress creates a layer of low PV, and frontal instabilities and meanders form ITEs. Their ζ is $\sim -f$ so that PV is approximately zero. The length scale of ITEs are belong to submesoscale, $\sim$ 6 km in radius, smaller compared to local internal Rossby radius of 7 km. The structure of induced ITEs can be expressed well by the model of [4].

The PV budget on the ITE isopycnal surface reveal the process we discussed above. From his Fig. 7 - 8, they represent an idea that before the wind shut down, the frictional forces keep extracting PV out of the front, where forms a region of low PV water. Once the wind is turned off, the PV flux reverses sign which indicates the front is slumping and high PV water are upwelled and advected into the front.

The configuration of the front in the model makes it initially baroclinically low PV. His Fig. 9 presents that the low PV is induced by the tilting effect because of meandering and vertical motions of ASCs (provide current shear). These effects not only reduce stratification but also favor the production of anticyclonic vorticity for $\zeta \sim -f$. Therefore, the interior of ITE change from baroclinically low PV to vortically low PV.

The paper finally examines how ITE interacts with the background flow field. I believe the author wants to reveal an idea that the loss of PV in isopycnal surfaces because ITEs also provide a positive eddy-induced PV flux. This can be readily seen in his Fig. 12 and 13. The temporal evolution of positive eddy-induced PV flux approximately match that of PV loss in the isopycnal layer. This indicates that ITE can advects PV.

2.4 Hypothesis of the divergent mode revealed by HFR data

Based on patterns using self-organizing map (SOM) revealed from HFR measurements in 2010, we found an vortex pair or the so-called ”divergent mode” seemed to appear during the wind relaxation period. The boundary that separates the two vortices likely located north of Icy Cape. The frequency of occurrence of this special pattern only occupied about 5% in 45-day long observation if twelve SOM patterns are used. The wind relaxation period is short, approximately about 2 day, then the southwesterly wind reduced and changed to the northeasterly wind.

The hypothesis is to treat the radar mask being separated into a more baroclinic northern portion and a more barotropic southern portion (**Fig. 6**). The meaning of more baroclinic means there will be more frontal structures and stronger stratification in the northern portion. The rationale of this assumption needs to be validated by hydrographic measurements.

If these fronts are surface trapped and outcropped, the wind stress provides the frictional forces and acts with the buoyancy gradient as a PV sink via the upward PV flux. Under this scenario, the northern part is spin-down ($\zeta \rightarrow -f$), whereas the southern part is spin-up because the fluid gain the input of wind curl ($\bigtriangledown \times \mathbf{F}$). When the wind subsides, the slumping front starts to receive high PV water which are upwelled below or advected from ambient water column. During the relaxation, the northern part turns to spin-up, but the south becomes spin-down and consequently develop a pair of vortices, which consist of a northern cyclone and a southern anticyclone. Same idea should be able to explain when the winds transit from northeasterly to southwesterly. Since the wind-driven PV destruction does not need the wind stress to be oriented in the down-front direction as mentioned in the paper ([6]).

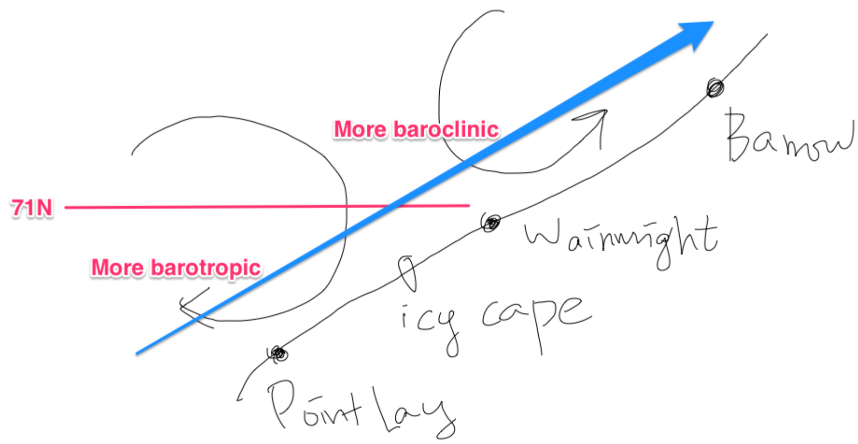

Figure 6: A cartoon representing the mechanism of divergent mode observed in the northeastern Chukchi Sea HFR mask. The blue arrow indicates the prevailing southwesterly wind. The latitude of 71^oN is presented indicating an imaginary boundary between the more baroclinic north and the more barotropic south.

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