Nore for deriving quasi-geostrophic potential vorticity and its rigid boundary conditions

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In stratified fluids, if flow structure contains a mean field and perturbation field then instability waves may be evolved by taking energy from the mean field to the perturbed field. The basic theory is the quasi-geostrophic (QG) model with stratification. In this document I recorded procedures of how to deriving the **boundary conditions** of perturbed potential vorticity field.

Using Vallis' book (2006) and following his notations (p.263), the conservation of potential vorticity is

$$\frac{\partial q}{\partial t} + \vec{u} \cdot \bigtriangledown q = 0 \ , 0 < z < H \tag{1}$$

where $q = Q + q'$ and Q is the mean field and q' is the perturbed field. If the upper and lower boundaries are treated as rigid (sea surface and bottom) then there is no vertical velocity. Therefore, the density field ρ is unchanged along these boundaries,

$$\frac{\partial \rho}{\partial t} + \vec{u} \cdot \bigtriangledown \rho = 0 \ , z = 0, H \tag{2}$$

If we multiply $-g/\rho_0$ to Eq. (2) and define $b = -\frac{g}{\rho_0}\rho$ then Eq. (2) becomes

$$\frac{\partial b}{\partial t} + \vec{u} \cdot \bigtriangledown b = 0 \ , z = 0, H \tag{3}$$

Th variable b is also called **buoyency**, it is often seen in some theoretical paper. Reviewing the notes for chapter 15 in Chusman Roisin's book about QG dynamics we can find that

$$\frac{\partial p}{\partial z} = -\rho g \tag{4}$$

and $p = \rho_0 f_0 \psi$. This is because the geostrophy is still dominant so the pressure can be expressed as a streamfunction. Therefore, $\rho = -\frac{1}{g}\frac{\partial p}{\partial z} = -\frac{1}{g}\rho_0 f_0 \frac{\partial \psi}{\partial z}$ and we can have

$$b = f_0 \frac{\partial \psi}{\partial z} \tag{5}$$

Then Eq. (3) becomes

$$\frac{\partial}{\partial t}\left(\frac{\partial \psi}{\partial z}\right) + \vec{u} \cdot \bigtriangledown \left(\frac{\partial \psi}{\partial z}\right) = 0 \ , z = 0, H \tag{6}$$

Next, we focus on the perturbed field, which can be thought as $q = Q + q'$, $u = U + u'$, ...etc.. We define the mean flow is along x direction and y direction is cross-stream direction so the potential vorticity in the perturbed field is,

$$\frac{\partial q'}{\partial t} + U\frac{\partial q'}{\partial x} + v'\frac{\partial Q}{\partial y} = 0 \tag{7}$$

where

$$Q = \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial}{\partial z}\left(\frac{{f_0}^2}{N^2}\frac{\partial \psi}{\partial z}\right) + \beta y \tag{8}$$

$$q' = \bigtriangledown^2 \psi' + \frac{\partial}{\partial z}\left(\frac{{f_0}^2}{N^2}\frac{\partial \psi'}{\partial z}\right) \tag{9}$$

Similarly, the perturbed density field along boundaries is (we do sea surface $z = H$ first)

$$\frac{\partial b'}{\partial t} + U\frac{\partial b'}{\partial x} + v'\frac{\partial b}{\partial y} = 0 \ , z = H \tag{10}$$

where express b' as $f_0 \frac{\partial \psi'}{\partial z}$ and so is for b,

$$\frac{\partial}{\partial t}\left(f_0 \frac{\partial \psi'}{\partial z}\right) + U\frac{\partial}{\partial x}\left(f_0 \frac{\partial \psi'}{\partial z}\right) + v'\frac{\partial}{\partial y}\left(f_0 \frac{\partial \psi}{\partial z}\right) = 0 \tag{11}$$

where on the third term if we do y-derivative first then we will have $\frac{\partial \psi}{\partial y} = -U$. Therefore Eq. (11) becomes

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\frac{\partial \psi'}{\partial z} - v'\frac{\partial U}{\partial z} = 0 \tag{12}$$

This can be seen in Johns (1988)'s eq(4) (at p.326)

If at sloping bottom (cross-stream direction), then we expect there will be vertical velocity advected by slope. The perturbed density field at the bottom ($z = 0$) will be

$$\frac{\partial b'}{\partial t} + U\frac{\partial b'}{\partial x} + v'\frac{\partial b}{\partial y} + w'\frac{\partial b}{\partial z} = 0 \ , z = 0 \tag{13}$$

Note that vertical velocity is induced by $w' = v'\frac{\partial h}{\partial y}$ and using the relation $\frac{\partial b}{\partial z} = N^2$, we will get

$$w'\frac{\partial b}{\partial z} = v'N^2\frac{\partial h}{\partial y} \tag{14}$$

Eq. (13) can be written as

$$\frac{\partial}{\partial t}\left(f_0\frac{\partial \psi'}{\partial z}\right) + U\frac{\partial}{\partial x}\left(f_0\frac{\partial \psi'}{\partial z}\right) + v'\frac{\partial}{\partial y}\left(f_0\frac{\partial \psi}{\partial z}\right) + v'N^2\frac{\partial h}{\partial y} = 0 \tag{15}$$

Using the relation $\frac{\partial \psi}{\partial y} = -U$ again and $v' = \frac{\partial \psi'}{\partial x}$, we will have

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\frac{\partial \psi'}{\partial z} = \frac{\partial \psi'}{\partial x}\frac{\partial U}{\partial z}(1 - {h_y}^{\star}) \ , z = 0 \tag{16}$$

where ${h_y}^{\star} = \left(\frac{\frac{\partial h}{\partial y}N^2}{f_0\frac{\partial U}{\partial z}}\right)$, is the ratio of bottom slope to the slope of the isopycnals. This can be seen in Johns (1988)'s eq(5) (at p.326)